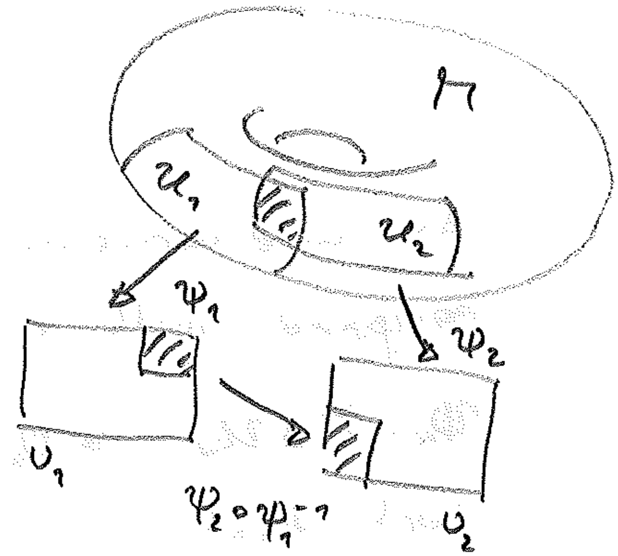
Reminder of Abhay's last lecture

I. Differential geometry



1. Manifolds

Some basic definitions

(i) Let M be a set, a covering $\mathcal{C}$ of M is a collection $\mathcal{U}_1, \dots, \mathcal{U}_N$, $N < \infty$ of subsets of M such that $\forall p \in M \ \exists i: p \in \mathcal{U}_i$.

(ii) A coordinate system is a 1-1 map ψ_i from $\mathcal{U}_i$ to an open subset U_i of $\mathbb{R}^n$.

(iii) Two charts ψ_1, ψ_2 are called compatible if the transition functions

$$\psi_1 \circ \psi_2^{-1} : \psi_1(\mathcal{U}_1 \cap \mathcal{U}_2) \to \mathbb{R}^n$$
$$\psi_2 \circ \psi_1^{-1} : \psi_2(\mathcal{U}_1 \cap \mathcal{U}_2) \to \mathbb{R}^n$$

are smooth (i.e. C^∞). Replacing here C^∞ by C^k may be sometimes necessary.

(iv) We call any subset $\mathcal{U} \subset \mathcal{U}_i$ open if $\psi_i(\mathcal{U})$ is open in $\mathbb{R}^n$.

With these preparations we are now ready to define a smooth n-dimensional manifold M:

An n-dimensional manifold M is a set equipped with a collection of charts $(\mathcal{U}_\alpha, V_\alpha, \psi_\alpha)$
$\psi_\alpha : M \supset \mathcal{U}_\alpha \longrightarrow V_\alpha \subset \mathbb{R}^n \; ; \; p \mapsto \psi_\alpha(p) = (x^1(p), \dots, x^n(p))$
such that:

(i) Any two charts are compatible.

(ii) $M = \bigcup_\alpha \mathcal{U}_\alpha$

(iii) Any chart which is compatible with all the other charts is itself in the collection.

(iv) If $p, q \in M : p \neq q$ then there exist open neighbourhoods $\mathcal{U}_p$, $\mathcal{U}_q$ of p and q respectively such that $\mathcal{U}_p \cap \mathcal{U}_q \neq 0$
(where $\mathcal{U}_p$ is an open neighbourhood of p, if $\mathcal{U}_p$ is open and $p \in \mathcal{U}_p$)

Point (iv) is the Hausdorff seperation axiom.

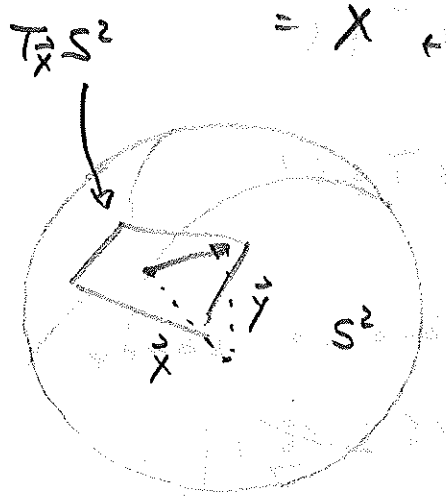
2. Tangent vectors and tensors on M

2.1 Motivation

Consider the unit sphere $S^2 = \{\vec{x} \in \mathbb{R}^3 : |\vec{x}| = 1\}$
The tangent space at a point $\vec{x} \in S^2$ is the 2-dimensional plane:

$$\{\vec{y} \in \mathbb{R}^3 \mid \vec{y} \cdot \vec{x} = 1\} =$$

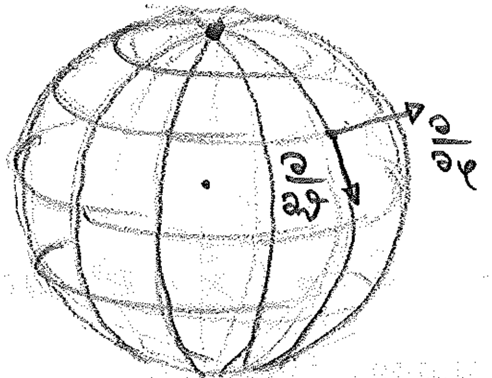
$$= \vec{x} + \underbrace{\{\vec{y} \in \mathbb{R}^3 \mid \vec{x} \cdot \vec{y} = 0\}}$$



This is a 2-dimensional vectorspace $T_{\vec{x}}S^2$.

This is, however, not a very useful construction – it works only because we have embedded S^2 into $\mathbb{R}^3$. The question is, therefore: Can we describe $T_{\vec{x}}S^2$ through structures intrinsically defined on S^2?

Idea: Look at smooth curves meeting in a point.



Consider the following parametrization of the sphere:

$\vartheta \in (0,\pi)$; $\varphi \in [0, 2\pi)$

$$\vec{X}(\vartheta,\varphi) = \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix} \in S^2 - \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Compute the velocities:

$$\frac{\partial}{\partial\vartheta}\vec{X} = \begin{pmatrix} \cos\vartheta \cos\varphi \\ \cos\vartheta \sin\varphi \\ -\sin\vartheta \end{pmatrix} = \vec{V}_\vartheta\big|_{\vec{X}} \in T_{\vec{X}}S^2$$

$$\frac{\partial}{\partial\varphi}\vec{X} = \sin\vartheta \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \vec{V}_\varphi\big|_{\vec{X}} \in T_{\vec{X}}S^2$$

Now, $\vec{V}_\vartheta\big|_{\vec{X}}$ and $\vec{V}_\varphi\big|_{\vec{X}}$ span the tangent space $T_{\vec{X}}S^2$ at $\vec{X} = \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix} \in S^2 - \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}$.

The idea is now to identify the tangent vectors $\vec{V}_\vartheta$ and $\vec{V}_\varphi$ with the directional derivatives $\frac{\partial}{\partial\vartheta}$ and $\frac{\partial}{\partial\varphi}$ on the sphere, thus removing the embedding manifold $\mathbb{R}^3$ from the definition of $T_{\vec{X}}S^2 = \operatorname{span}\{\vec{V}_\vartheta|_{\vec{X}}, \vec{V}_\varphi|_{\vec{X}}\}$.

2.2 Tangent vectors

We can now formalize the intuition

$$\boxed{\text{tangent vectors} = \text{directional derivatives}}$$

Def: Tangent vector at a point $p \in M$.

Preparation: We call a function $f: M \to \mathbb{R}^m$ smooth; i.e. $f \in C^\infty(M:\mathbb{R}^m)$ if for any chart $(\mathcal{U}, U, \psi)$ the function

$$\tilde{f} = (f \circ \psi^{-1}) : \underset{\mathbb{R}^n}{\underset{\cap}{\psi(U)}} \to \mathbb{R}^m \quad \text{is smooth}$$

Tangent vector: A function $V_p : C^\infty(M:\mathbb{R}) \to \mathbb{R}$ is said to be a tangent vector (or directional derivative) at $p \in M$ (symbolically $V_p \in T_pM$) if the following two conditions hold true:

(i. Linearity) $\forall\, f, g \in C^\infty(M:\mathbb{R})$; $a, b \in \mathbb{R}$:

$$V_p[af + bg] = a V_p[f] + b V_p[g]$$

(ii. Leibniz rule) $\forall\, f, g \in C^\infty(M:\mathbb{R})$:

$$V_p[f \cdot g] = V_p[f]\, g(p) + f(p)\, V_p[g]$$

This further implies:

(i) For the constant function $1: M \to \mathbb{R},\ p \mapsto 1$

$$V_p[1] = 0$$

(ii) There's the null vector

$$\mathbb{O}: C^\infty(M) \to \mathbb{R};\ f \mapsto \mathbb{O}[f] = 0$$

clearly $\mathbb{O} \in T_pM \quad \forall\ p \in M$.

(iii) If V_p and $U_p \in T_pM$ then also $aV_p + bU_p \in T_pM$, for all $a, b \in \mathbb{R}$, hence T_pM is a real vector space.

<u>Examples for tangent vectors</u>

(1) Let $\gamma: [-1,1] \to \mathcal{U} \subset M$ be a smooth path through $\gamma(0) = p$; The map

$$\dot{\gamma}_p[f] = \frac{d}{dt}\Big|_{t=0} (f \circ \gamma)(t) \quad \text{for all } f \in C^\infty(M:\mathbb{R})$$

fulfills all requirements, thus $\dot{\gamma}_p \in T_pM$. In a local chart $(\mathcal{U}, U, \psi)$ γ turns into a path $(\psi \circ \gamma): [-1,1] \to U;\ t \mapsto (\psi \circ \gamma)(t) = (x^1(\gamma(t)), \dots, x^n(\gamma(t)))$; hence:

$$\dot{\gamma}_p[x^\mu] = \frac{d}{dt}\Big|_{t=0} x^\mu(\gamma(t)) \ ; \quad \mu = 1, \dots, n$$

(2) Let $(V^1, \dots, V^n) \in \mathbb{R}^n$, consider a chart $(\mathcal{U}, \mathcal{V}, \psi)$ and define for any $f \in \mathcal{C}^\infty(M:\mathbb{R})$:

$$\tilde{f}(x^1, \dots, x^n) = (f \circ \psi^{-1})(x^1, \dots, x^n)$$

$$V_p[f] = \sum_{\mu=1}^{n} V^\mu \frac{\partial \tilde{f}}{\partial x^\mu}\Big|_{\psi(p)}$$

then $V_p \in T_pM$. In fact, any $V_p \in T_pM$ can be written in this form.

It now follows:

(i) T_pM has dimensions $\dim(M) = \dim(T_pM) = n$.

(ii) If $(\mathcal{U}, \mathcal{V}, \psi)$ is a chart around $p \in \mathcal{U}$ then the derivatives

$$X_\mu\big|_p [f] = \frac{\partial (f \circ \psi^{-1})}{\partial x^\mu}\Big|_{\psi(p)} \quad ; \ \forall f \in \mathcal{C}^\infty(M:\mathbb{R})$$

with $\mu = 1, \dots, n$ form a basis in T_pM.

We will now prove this by explicitly constructing a basis in T_pM:

Consider first a smooth function $\tilde{f} \in C^\infty(M;\mathbb{R})$ with $U \subset \mathbb{R}^n$; we write:

$$\tilde{f}(x) - \tilde{f}(x_0) = \int_0^1 dt \, \frac{d}{dt} \tilde{f}(tx + (1-t)x_0) =$$

$$\underset{\text{chain rule}}{=} \sum_{\mu=1}^{n} (x^\mu - x_0^\mu) \int_0^1 dt \, \frac{\partial \tilde{f}}{\partial x^\mu}\bigg|_{tx+(1-t)x_0}$$

We now lift this equation to the manifold. Let $(\mathcal{U}, U, \psi)$ be a chart; for any $p, p_0 \in \mathcal{U}$ and $\tilde{f} = f \circ \psi^{-1}$ for some $f \in C^\infty(M;\mathbb{R})$:

$$\boxed{f(p) - f(p_0) = \sum_{\mu=1}^{n} (x^\mu(p) - x^\mu(p_0)) \int_0^1 dt \, \frac{\partial (f \circ \psi^{-1})}{\partial x^\mu}\bigg|_{t\psi(p) + (1-t)\psi(p_0)}}$$

with $\psi(p) = (x^1(p), \dots, x^n(p))$

$$x^\mu(p) \equiv (x^\mu \circ \psi)(p)$$

$$f(p) = f(p_0) + \sum_{\mu=1}^{n} \left(x^\mu(p) - x^\mu(p_0)\right) \int_0^1 dt \; \frac{\partial(f \circ \psi^{-1})}{\partial x^\mu}\Big|_{t\psi(p) + (1-t)\psi(p_0)}$$

We now compute $V_p[f]$ while keeping p_0 fixed:

$$V_{p_0}[f] = \underbrace{V_{p_0}[f(p_0)]}_{=0} +$$

$$+ \sum_{\mu=1}^{n} V_{p_0}\left[x^\mu + x^\mu(p_0)\right] \int_0^1 dt \, \frac{\partial(f \circ \psi^{-1})}{\partial x^\mu}\Big|_{\psi(p_0)} +$$

$$+ \sum_{\mu=1}^{n} \underbrace{\left(x^\mu(p_0) - x^\mu(p_0)\right)}_{=0} V_{p_0}\left[\int_0^1 dt \; \frac{\partial(f \circ \psi^{-1})}{\partial x^\mu}\Big|_{t\psi(\cdot) + (1-t)\psi(p_0)}\right]$$

$$= \sum_{\mu=1}^{n} V_{p_0}[x^\mu] \;\; \frac{\partial(f \circ \psi^{-1})}{\partial x^\mu}\Big|_{\psi(p_0)}$$

For arbitrary $f \in C^\infty(M:\mathbb{R})$: $V_p[f] = 0 \iff V_p[x^\mu] = 0$.

Thus:

(i) T_pM has dimensions $\dim(T_pM) = \dim(M) = n$

(ii) If $(\mathcal{U}, U, \psi)$ is a chart, then the directional derivatives:

$$X_\mu|_p[f] = \frac{\partial(f \circ \psi^{-1})}{\partial x^\mu}\Big|_{\psi(p)} \; ; \quad \psi = (x^1, \dots, x^n)$$

form a basis in T_pM.

We can thus write:

$$V_p = \sum_{\mu=1}^{n} V_p[x^\mu]\, X_\mu|_p = \sum_{\mu=1}^{n} V_p^\mu\, X_\mu|_p$$

and we call $V_p[x^\mu] = V_p^\mu$ the components of $V_p \in T_pM$ with respect to the chart $(\mathcal{U}, \mathcal{O}, \psi)$; $p \in \mathcal{U}$; $\psi(p) = (x^1(p), \dots, x^n(p))$.

<u>Simplified notation</u>

- $X_\mu|_p[f] = \dfrac{\partial(f \circ \psi^{-1})}{\partial x^\mu}\Big|_p = \dfrac{\partial f}{\partial x^\mu}\Big|_p$
- $V_p[x^\mu \circ \psi] = V_p[x^\mu] = V_p^\mu$
- <u>Einstein Summation convention</u>

$$\sum_{\mu=1}^{n} V_p[x^\mu]\, X_\mu|_p = V_p^\mu\, X_\mu|_p = V_p^\mu \frac{\partial}{\partial x^\mu}\Big|_p$$

2.3 Coordinate Transformations

For a given chart $(\mathcal{U}, V, \varphi)$ we call the n-tuple

$$(X_1|_p, \dots, X_n|_p) = \left(\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right)$$

the coordinate basis of T_pM w.r.t. $(\mathcal{U}, V, \varphi)$.

What happens if we had chosen another overlapping chart $(\tilde{\mathcal{U}}, \tilde{V}, \tilde{\varphi})$; $\tilde{\mathcal{U}} \cap \mathcal{U} \neq \emptyset$?

Let $(\mathcal{U}, V, \varphi)$ and $(\tilde{\mathcal{U}}, \tilde{V}, \tilde{\varphi})$ be two charts with a common overlap $\mathcal{U} \cap \tilde{\mathcal{U}} \ni p$.

We write:

$$\varphi \circ \tilde{\varphi}^{-1} \equiv (x^1(\tilde{x}), \dots, x^n(\tilde{x}))$$
$$\tilde{\varphi} \circ \varphi^{-1} \equiv (\tilde{x}^1(x), \dots, \tilde{x}^n(x))$$

And then have:

$$V_p[f] = V_p^\mu \frac{\partial f}{\partial x^\mu}\Big|_p = V_p^\mu \frac{\partial \tilde{x}^\nu(x)}{\partial x^\mu}\Big|_p \frac{\partial f}{\partial \tilde{x}^\nu}\Big|_p =$$

$$= \tilde{V}_p^\mu \frac{\partial f}{\partial \tilde{x}^\mu}\Big|_p$$

<u>thus</u>.

$$\boxed{\begin{aligned} \tilde{V}_p^\mu &= V_p^\mu \frac{\partial \tilde{x}^\nu(x)}{\partial x^\mu} \\ \frac{\partial}{\partial x^\mu}\Big|_p &= \frac{\partial \tilde{x}^\nu(x)}{\partial x^\mu} \frac{\partial}{\partial \tilde{x}^\nu}\Big|_p \end{aligned}}$$

Where $J^\mu{}_\nu(x) = \dfrac{\partial \tilde{x}^\mu(x)}{\partial x^\nu}$ is the Jacobi matrix.

$$(J^{-1})^\mu{}_\nu(\tilde{x}) = \frac{\partial \tilde{x}^\mu(x)}{\partial x^\nu}$$

$$J^\mu{}_\nu (J^{-1})^\nu{}_\alpha = \delta^\mu_\alpha$$