Reminder of our last lecture

Tangent vectors

A tangent vector $V_p \in T_pM$ is a map $V_p : C^\infty(M:\mathbb{R}) \longrightarrow \mathbb{R}$ that fulfills for all $a, b \in \mathbb{R}$ and $f, g \in C^\infty(M:\mathbb{R})$ that:

(i. Linearity)

$$V_p[af + bg] = aV_p[f] + bV_p[g]$$

(ii. Leibniz rule)

$$V_p[f \cdot g] = V_p[f]\, g(p) + f(p)\, V_p[g]$$

We also saw:

(i.) T_pM is a real vector space of dimension $\dim(T_pM) = \dim(M) = n$

(ii.) For any smooth path $\gamma : [-1,1] \to M ; t \mapsto \gamma(t)$ passing through $p = \gamma(0)$ the equation

$$\dot{\gamma}_p[f] = \frac{d}{dt}\Big|_{t=0} (f \circ \gamma)(t)$$

defines a tangent vector

(iii.) If $\psi : \mathcal{U} \to U$ is a local chart with $\psi(p) = (x^1(p), \ldots, x^n(p))$ then the tangent vectors $(X_1, \ldots, X_n)$ defined for any $f \in C^\infty(M : \mathbb{R})$ through the equation:

$$X_\mu [f] \Big|_p = \frac{\partial (f \circ \psi^{-1})}{\partial x^\mu} \Big|_p$$

form a basis - the coordinate basis - in $T_p M$.

Outline of today's lecture

2.3 Vector fields - the tangent bundle

2.4 Covector fields - the cotangent bundle

2.5 Tensors

From now on, we will use a simplifying notation:

- $X_\mu|_p [f] = \frac{\partial (f \circ \varphi^{-1})}{\partial x^\mu}\Big|_{\varphi(p)} = \frac{\partial f}{\partial x^\mu}\Big|_p$
- $V_p[x^\mu] \equiv V_p[x^\mu \circ \varphi] = V_p^\mu$
- Einstein summation convention

$$\sum_{\mu=1}^{n} V_p[x^\mu] X_\mu|_p = V_p^\mu \frac{\partial}{\partial x^\mu}\Big|_p$$

2.3 Vector fields - the tangent bundle

tangent bundle: We call the union

$TM := \bigcup_{p \in M} \{p\} \times T_pM$ the tangent bundle of M.

vector fields We call a map

$V: M \to TM;\ p \mapsto T_pM$ a smooth vector field (symbolically denoted by $V \in \Gamma(TM)$) if $V[f]: M \to \mathbb{R};\ p \mapsto V_p[f]$ is a smooth function for all $f \in C^\infty(M;\mathbb{R})$

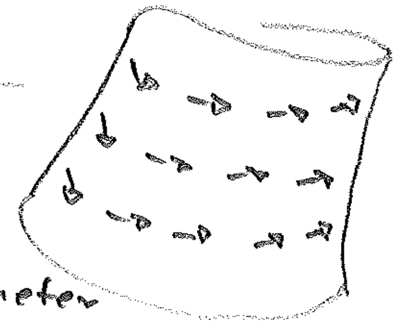
Now, the directional derivatives $X_\mu = \frac{\partial}{\partial x^\mu}$ are already smooth vectorfields; the form a basis hence:

> V is a smooth vectorfield if (for a collection of charts covering all of M) its component functions $V^\mu = V[x^\mu]$ are smooth functions $V[x^\mu]: M \ni p \mapsto V_p[x^\mu] \in \mathbb{R}$ $\forall \mu = 1, \dots, n$ themselves

The geometric interpretation of vectorfields

Vector fields define flows on the manifold

a flow defines a vectorfield



A flow is a smooth 1-parameter family $\phi_t : M \to M$ of diffeomorphisms:

(i) ϕ_t is one-to-one $\forall t \in \mathbb{R}$

(ii) $\phi_t \in C^\infty(M:M)$

(iii) $\phi_t^{-1} \in C^\infty(M:M)$

If we pick a point $p \in M$; $t \mapsto \phi_t(p)$ defines a smooth curve, and we can then define a smooth vectorfield V through:

$$V_p = \frac{d}{dt}\Big|_{t=0} \phi_t(p) \in T_p M$$

i.e.

$$V_p[f] = \frac{d}{dt}\Big|_{t=0} (f \circ \phi_t)(p) \quad \forall \; f \in C^\infty(M, \mathbb{R})$$

the opposite is also true:
a vectorfield defines a flow

Given a vectorfield V we can always find its integral curves in M: We have to solve the following differential equation in every chart $(\mathcal{U}, U, \psi)$ covering M:

$$\frac{d}{dt} x^\mu(t) = V^\mu(x^1(t), \ldots, x^n(t)) \quad \forall \; \mu = 1, \ldots, n$$

$$(x^1(0), \ldots, x^n(0)) = p$$

and we then set:

$$\phi_t(p) = \psi^{-1}(x^1(t), \ldots, x^n(t))$$

We can solve this differential equation iteratively, and thus write:

$$\boxed{\phi_t = \exp(t V)}$$

Given two smooth vectorfields U, V we can define the Lie-Bracket:

$\forall f \in C^\infty(M:\mathbb{R})$:

$$[U,V]_p[f] = U_p[V.[f]] - V_p[U.[f]]$$

For any three smooth vectorfields we always have:

$$[[U,V],W] + [[V,W],U] + [[W,U],V] = 0$$

Given two commuting vectorfields U, V: $[U,V]=0$ we can always find a two-parameter family of diffeomorphisms $\phi_{s,t}$ such that:

$$U_p = \frac{d}{ds}\Big|_{s,t=0} \phi_{s,t}(p) \in T_pM$$

$$V_p = \frac{d}{dt}\Big|_{s,t=0} \phi_{s,t}(p) \in T_pM$$

2.4 covector fields - the cotangent bundle

Given a (real) vectorspace $\mathbb{V}$ we can always build the dual vectorspace $\mathbb{V}^*$ as the space of all linear maps:

$$\omega : \mathbb{V} \to \mathbb{R}; \quad V \mapsto \omega(V)$$

We have:

(i) $\forall\, a, b \in \mathbb{R}$; $U, V \in \mathbb{V}$; $\omega \in \mathbb{V}^*$

$$\omega(aU + bV) = a\,\omega(U) + b\,\omega(V)$$

(ii) If e_μ, $\mu = 1, \dots, n$ is a basis in $\mathbb{V}$ then the equation

$$\tilde{e}^\mu(e_\nu) = \delta^\mu_\nu, \quad \tilde{e}^\mu \in \mathbb{V}^*$$

defines a basis in $\mathbb{V}^*$.

<u>hence</u>:

$$\dim(\mathbb{V}) = \dim(\mathbb{V}^*)$$

(iii) $(\mathbb{V}^*)^* \simeq \mathbb{V}$

At each point p of the manifold we have the tangent vector space T_pM. We now build its dual vector space T_p^*M and call it the cotangent space T_p^*M.

We call the elements ω_p of T_p^*M co(variant) vectors.

Co-vector bundle

- We call the union $T^*M = \bigcup_{p \in M} \{p\} \times T_p^*M$ the cotangent bundle of M.
- A map $\omega : M \to T^*M$ is a smooth co vector field if for all smooth vector fields V the function $\omega(V) : M \to \mathbb{R}$; $p \mapsto \omega_p(V)$ is smooth. We then write $\omega \in \Gamma(T^*M)$

Example of a smooth co-vector field

Let $f \in C^\infty(M : \mathbb{R})$ be a smooth function, we define its differential df_p at the point $p \in M$ by declaring for all tangent vectors $V_p \in T_pM$:

$$(df)_p(V) = V_p[f]$$

Now, for all $a, b \in \mathbb{R}$; $U, V \in T_pM$:

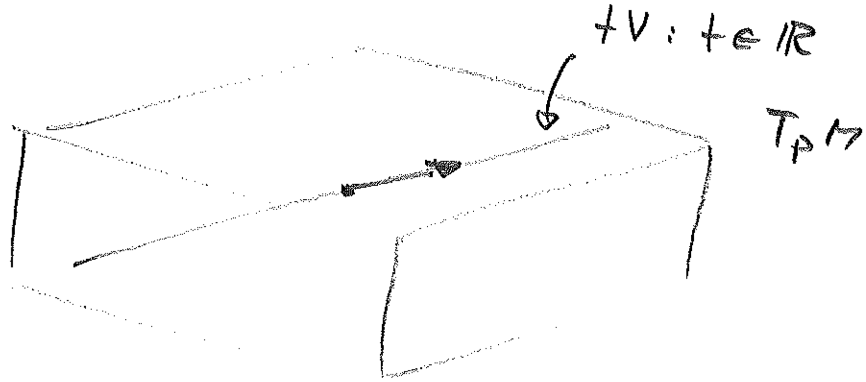
$$(df)_p(aU + bV) = a(df)_p(U) + b(df)_p(V)$$

Hence

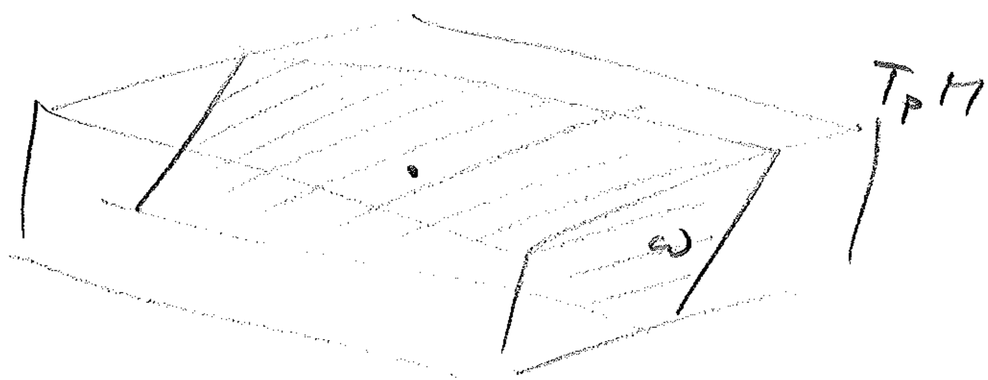
$$\boxed{df_p \in T^*_pM}$$

The geometric interpretation of co-vectors

- A vector $V \in T_pM$ defines a one-dimensional line in T_pM:



- A co-vector $\omega \in T_p^*M$ defines a $(n-1)$-dimensional hypersurface $\{V \in T_pM \mid \omega(V) = 0\}$ in T_pM:



Nota bene

- Vectorfields can always be integrated

$$V \longrightarrow \exp(tV)$$

- For covectors this is not true:

$$\omega \in T^*M \text{ in general } \nexists f : \omega = df$$

A basis in $T_p^* M$

We have already seen that given a chart $(\mathcal{U}, V, \psi)$ around $p \in \mathcal{U}$; $\psi(p) = (x^1(p), \dots, x^n(p))$ the derivatives:

$$(X_1|_p, \dots, X_n|_p) = \left(\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right)$$

form a basis in $T_p M$.

We now use them to construct a basis in $T_p^* M$:

Consider the differentials of the coordinate functions x^μ $\mu = 1, \dots, n$ around p:

$$(dx^\mu)_p (X_\nu) = X_\nu|_p [x^\mu] = \frac{\partial x^\mu}{\partial x^\nu}\Big|_p = \delta^\mu_\nu$$

We now set for any $\omega \in T_p^* M$:

$$\omega_\mu := \omega\left(\frac{\partial}{\partial x^\mu}\right)$$

and call ω_μ the μ-th component of ω w.r.t. the chart $(\mathcal{U}, U, \varphi)$ around p.

We now have:

$$\boxed{\omega = \omega_\mu \, dx^\mu}$$

Which can be seen as follows:

$$\omega(V) = \omega\left(V^\mu \frac{\partial}{\partial x^\mu}\right) = V^\mu \omega\left(\frac{\partial}{\partial x^\mu}\right) = V^\mu \omega_\mu$$

$$(\omega_\mu \, dx^\mu)(V) = \omega_\mu \, (dx^\mu)(V) = \omega_\mu \, V[x^\mu] = \omega_\mu V^\mu$$

Hence

$$\dim(T_p M) = \dim(T_p^* M) = n.$$

2.5 Tensor fields

- Given a (real) vectorspace $\mathbb{V}$ we saw how to construct its dual vector space $\mathbb{V}^*$.
- We naturally have $(\mathbb{V}^*)^* = \mathbb{V}$.
- We can now also build tensors of rank (r,s):

A tensor T of rank (r,s) is a multilinear map $T: \underbrace{\mathbb{V}^* \times \ldots \times \mathbb{V}^*}_{r\text{-times}} \times \underbrace{\mathbb{V} \times \ldots \times \mathbb{V}}_{s\text{-times}} \to \mathbb{R}$.

Basic properties

(i) The space of all tensors of rank (r,s) is a real vectorspace $\mathbb{V}^r{}_s$:

if $T, S \in \mathbb{V}^r{}_s$; $a, b \in \mathbb{R}$

$$aT + bS \in \mathbb{V}^r{}_s$$

(ii) We naturally have

$$\mathbb{V}^0{}_1 = \mathbb{V}^*$$

$$\mathbb{V}^1{}_0 = \mathbb{V}$$

(iii) $\dim(\mathbb{V}^r{}_s) = n^{r+s}$

Tensor product

If S is a tensor of rank (r,s) and T is a tensor of rank (l,m) we can build a tensor $(T \otimes S)$ of rank $(r+l, s+m)$ by declaring:

$$(S \otimes T)(\omega_1, \dots, \omega_{r+l}, V_1, \dots, V_{s+m}) :=$$

$$= S(\omega_1, \dots, \omega_r, V_1, \dots, V_s)\, T(\omega_{r+1}, \dots, \omega_{r+l}, V_{r+1}, \dots, V_{r+l})$$

Basic properties

(i) $a, b \in \mathbb{R}$; R, S, T tensors:

- $(aR + bS) \otimes T = a(R \otimes T) + b(S \otimes T)$
- $T \otimes (aR + bS) = a\, T \otimes R + b\, T \otimes S$
- $a\, T \otimes S = (aT) \otimes S = T \otimes (aS)$

(ii) S, T tensors

$$S \otimes T \neq T \otimes S$$

(iii) R, S, T tensors

$$(R \otimes S) \otimes T = R \otimes (S \otimes T) =$$
$$= R \otimes S \otimes T$$

(iv) If $\{e_\mu\}$ $\mu = 1, \dots, n$, $n = \dim(W)$ is a basis in W and $\{\tilde{e}^\mu\}$, $\mu = 1, \dots, n$ is the dual basis in W^* then

$$e_{\mu_1} \otimes \dots \otimes e_{\mu_r} \otimes \tilde{e}^{\nu_1} \otimes \dots \otimes \tilde{e}^{\nu_s}$$

is a basis for all tensors of rank (r, s), which follows immediately from:

$$(e_{\mu_1} \otimes \dots \otimes e_{\mu_r} \otimes \tilde{e}^{\nu_1} \otimes \dots \otimes \tilde{e}^{\nu_s})(\tilde{e}^{\alpha_1}, \dots, \tilde{e}^{\alpha_r}, e_{\beta_1}, \dots, e_{\beta_s}) =$$
$$= \delta^{\alpha_1}_{\mu_1} \cdot \ldots \cdot \delta^{\alpha_r}_{\mu_r} \, \delta^{\nu_1}_{\beta_1} \cdot \ldots \cdot \delta^{\nu_s}_{\beta_s}$$

We can thus write for any tensor of rank (r,s):

$$T = T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\, e_{\mu_1} \otimes \dots \otimes e_{\mu_r} \otimes \tilde{e}^{\nu_1} \otimes \dots \otimes \tilde{e}^{\nu_s}$$

And can now define the following operations

- Symmetrisation:

$$T_{(\mu_1 \dots \mu_k)} = \frac{1}{k!} \sum_{\pi \in S_k} T_{\mu_{\pi(1)} \dots \mu_{\pi(k)}}$$

- Anti-Symmetrisation:

$$T_{[\mu_1 \dots \mu_k]} = \frac{1}{k!} \sum_{\pi \in S_k} \mathrm{sign}(\pi)\, T_{\mu_{\pi(1)} \dots \mu_{\pi(k)}}$$

- <u>Contraction</u>

If T is a tensor of rank (r,s) we define the contraction $C^i{}_j$ over the i-th and j-th idices as the following operation:

$$C^i{}_j T = T^{\mu_1 \dots \overset{i\text{-th}}{\alpha} \dots \mu_r}{}_{\nu_1 \dots \underset{j\text{-th}}{\alpha} \dots \nu_s}$$

$$e_{\mu_1} \otimes \dots e_{\mu_{i-1}} \otimes e_{\mu_{i+1}} \otimes \dots e_{\mu_r} \otimes$$
$$\tilde{e}^{\nu_1} \otimes \dots \tilde{e}^{\nu_{j-1}} \otimes \tilde{e}^{\nu_{j+1}} \otimes \dots \tilde{e}^{\nu_s}$$

- $C^i{}_j T$ is a tensor of rank $(r-1, s-1)$

The result of the operations: <u>contraction</u>, <u>symmetrization</u> and <u>anti-symmetrization</u> does not depend on the actual basis chosen.